On Colorings of graph fractional powers

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Dedicated to Professor Cheryl Praeger
on her 60th birthday

Abstract

For any $k \in \mathbb{N}$, the $k-$subdivision of graph G is a simple graph $G^{\frac{1}{k}}$, which is constructed by replacing each edge of G with a path of length k. In this paper we introduce the mth power of the $n-$subdivision of G, as a fractional power of G, denoted by $G^{\frac{m}{n}}$. In this regard, we investigate chromatic number and clique number of fractional power of graphs. Also, we conjecture that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ provided that G is a connected graph with $\Delta(G) \geq 3$ and $\frac{m}{n} < 1$. It is also shown that this conjecture is true in some special cases.

Key words: Chromatic number, Subdivision of a graph, Power of a graph.

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1 Introduction and Preliminaries

In this paper we only consider simple graphs which are finite, undirected, with no loops or multiple edges. We mention some of the definitions which are referred to throughout the paper. As usual, we denote by $\Delta(G)$ or simply by Δ, the maximum degrees of the vertices of graph G, and by $\omega(G)$, the maximum size of a clique of G, where a clique of G is a set of mutually adjacent vertices. In addition $N_G(v)$, called the neighborhood of v in G, denotes the set of vertices of G which are adjacent to the vertex v of G. For another necessary definitions and notations we refer the reader to textbook [4].

Let G be a graph and k be a positive integer. The $k-$*power of* G, denoted by G^k, is defined on the vertex set $V(G)$, by connecting any two distinct vertices x and y with distance at most k [1,8]. In other words, $E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\}$. Also $k-$*subdivision of* G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge ij of G with a path of length k, say P_{ij}. These $k-$pathes are called *hyperedges* and any new vertex is called *internal vertex* or brifely *i-vertex* and is denoted by $i(l)j$, if it belongs to the hyperedge P_{ij} and has distance l from the vertex i, where $l \in \{1, 2, \ldots, k-1\}$. Note that $i(l)j = j(k-l)i$, $i(0)j \overset{\text{def}}{=} i$ and

$i(k)j \stackrel{\text{def}}{=} j$. Also any vertex $i = i(0)j$ of $G^{\frac{1}{k}}$ is called *terminal vertex* or brifely *t-vertex*.
Note that for $k = 1$, we have $G^{\frac{1}{1}} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges.
Now we can define the fractional power of a graph as follows.

Definition 1. Let G be a graph and $m, n \in \mathbb{N}$. The graph $G^{\frac{m}{n}}$ is defined by the $m-$power of the $n-$subdivision of G. In other words $G^{\frac{m}{n}} \stackrel{\text{def}}{=} (G^{\frac{1}{n}})^m$. ♠

Note that the graphs $(G^{\frac{1}{n}})^m$ and $(G^m)^{\frac{1}{n}}$ are different graphs and in this paper, we only consider the graphs $(G^{\frac{1}{n}})^m$ for all positive integers m and n. Furthermore, we assume that $V(G^{\frac{m}{n}}) = V(G^{\frac{1}{n}})$.
As you know, for any graph G, we have $\chi(G) \leq \chi(G^2) \leq \chi(G^3) \leq \ldots \leq \chi(G^d) = |V(G)|$, where d is the diameter of G. In other words, the chromatic number of a graph G increases when we replace G by its powers. The problem of coloring of graph powers, specially planar graph powers, is studied in the literature (see for example [1, 2, $5-11$]).
On the other hand, it is easy to verify that by replacing any graph by its subdivisions, the chromatic number of graph decreases. In other words, $\chi(G) \geq \chi(G^{\frac{1}{k}})$ for any $k \in \mathbb{N}$. Now the following question arises naturally.
Question. *What happened for the chromatic number, when we consider the fractional power of a graph, specially less than (or more than) one power?*
In this paper we answer to this question for the less than one power of a graph and find the chromatic number of $G^{\frac{m}{n}}$ for rational number $\frac{m}{n} < 1$.
Another motivation for us to define the fractional power of a graph is the *Total Coloring Conjecture*. Vizing (1964) and, independently, Behzad (1965) conjectured that the total chromatic number of a simple graph G never exceeds $\Delta+2$ (and thus $\Delta+1 \leq \chi''(G) \leq \Delta+2$) [3,12]. By virtue of Definition 1, the total chromatic number of a simple graph G is equal to $\chi(G^{\frac{2}{2}})$ and $\omega(G^{\frac{2}{2}}) = \Delta + 1$ when $\Delta \geq 2$. Thus, the Total Coloring Conjecture can be reformulated as follows.

Total Coloring Conjecture. *Let G be a simple graph and $\Delta(G) \geq 2$. Then $\chi(G^{\frac{2}{2}}) \leq \omega(G^{\frac{2}{2}}) + 1$.*

In the next section, at first, we calculate the clique number of $G^{\frac{m}{n}}$ when $\frac{m}{n} < 1$. Next, the chromatic number of $G^{\frac{m}{n}}$ is calculated when $\Delta(G) = 2$. Later, we show that $\chi(G^{\frac{2}{n}}) = \omega(G^{\frac{2}{n}})$ for any positive integer $n \geq 3$ when $\Delta(G) \geq 3$. Finally we show that $\chi(G^{\frac{m}{m+1}}) = \omega(G^{\frac{m}{m+1}})$ when G is a connected graph with $\Delta(G) \geq 3$ and $m \in \mathbb{N}$, which leads us to claim the following conjecture.
Conjecture A. *Let G be a connected graph with $\Delta(G) \geq 3$, $n, m \in \mathbb{N}$ and $1 < m < n$. Then*

$$\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}}).$$

2 Main Results

Here, we find the clique number of $G^{\frac{m}{n}}$ when $\frac{m}{n} < 1$.

Theorem 1. *Let G be a graph, $n, m \in \mathbb{N}$ and $m < n$. Then*

$$\omega(G^{\frac{m}{n}}) = \begin{cases} m+1 & \Delta(G) = 1 \\ \frac{m}{2}\Delta(G)+1 & \Delta(G) \geq 2, m \equiv 0\ (mod\, 2) \\ \frac{m-1}{2}\Delta(G)+2 & \Delta(G) \geq 2, m \equiv 1\ (mod\, 2). \end{cases}$$

Proof. We consider three cases:
Case 1. $\Delta(G) = 1$
In this case, each connected component of G is K_2 or K_1. Therefore, $\omega(G^{\frac{m}{n}}) = \omega(K_2^{\frac{m}{n}}) = \omega(P_{n+1}^m)$. It is clear that $\omega(P_{n+1}^m) = m+1$.
Case 2. $\Delta(G) \geq 2$, $m \equiv 0\ (mod\, 2)$
Choose a t-vertex x of $G^{\frac{m}{n}}$ such that $d_G(x) = \Delta(G)$. Consider the set $S = \{y : d_{G^{\frac{m}{n}}}(y,x) \leq \frac{m}{2}\}$. Obviously, this set is a clique with $\frac{m}{2}\Delta(G)+1$ vertices. Now, we show that any clique of $G^{\frac{m}{n}}$ has at most $\frac{m}{2}\Delta(G)+1$ vertices. Let C be a maximal clique of $G^{\frac{m}{n}}$. The distance of any two t-vertices in $G^{\frac{m}{n}}$ is at least two. Therefore, C has at most one t-vertex. First, suppose that C has not any t-vertex. Thus, all vertices of C belong to a hyperedge and hence $|C| \leq m+1 \leq \frac{m}{2}\Delta(G)+1$. Now, suppose that C has a t-vertex x. Let y has the greatest distance from x among the other vertices of C. If $d_{G^{\frac{m}{n}}}(y,x) = d > \frac{m}{2}$, then C has at most $m-d$ vertices in common with any hyperedge that is adjacent with x and doesn't contain y. Therefore,

$$\begin{aligned} |C| &= (d(x)-1)(m-d)+d+1 \leq (\Delta(G)-2)(m-d)+m+1 \\ &\leq (\Delta(G)-2)\frac{m}{2}+m+1 = \Delta(G)\frac{m}{2}+1. \end{aligned}$$

In other case, if $d \leq \frac{m}{2}$, then C has at most $\frac{m}{2}$ vertices in common with any hyperedge that is adjacent to x. So $|C| \leq \frac{m}{2}\Delta(G)+1$.
Case 3. $\Delta(G) \geq 2$, $m \equiv 1\ (mod\, 2)$
Choose a t-vertex x of $G^{\frac{m}{n}}$ such that $d_G(x) = \Delta(G)$. Consider the sets $S_1 = \{y : d_{G^{\frac{m}{n}}}(y,x) \leq \frac{m-1}{2}\}$ and $S_2 = \{z : d_{G^{\frac{m}{n}}}(z,x) = \frac{m+1}{2}\}$. Obviously each element of S_2 along with the vertices of S_1, form a clique of size $\frac{m-1}{2}\Delta(G)+2$. Now, similar to case 2, one can prove that any clique of $G^{\frac{m}{n}}$ has at most $\frac{m-1}{2}\Delta(G)+2$ vertices. ■

F. Kramer and H. Kramer gave in [9] the following characterization of the graphs for which $\chi(G^m) = m+1$.

Theorem A. [9] *Let $G = (V, E)$ be a simple and connected graph and $m \in \mathbb{N}$. We have $\chi(G^m) = m + 1$ if and only if the graph G is satisfying one of the following conditions:*
(a) $|V| = m + 1$,
(b) G is a path of length greater than m,
(c) G is a cycle of length a multiple of $m + 1$.

To find the chromatic number of $G^{\frac{m}{n}}$ in the case $\frac{m}{n} < 1$, we only consider the connected graphs in two cases $\Delta(G) = 2$ and $\Delta(G) > 2$. Theorem A can be considered as a result of Theorems 1 and 2.

Theorem 2. *Let $m, k \in \mathbb{N}$ and $k \geq 3$. Then*
(a) $\chi(C_k^m) = \begin{cases} k & m \geq \lfloor \frac{k}{2} \rfloor \\ \lceil \frac{k}{\lfloor \frac{k}{m+1} \rfloor} \rceil & m < \lfloor \frac{k}{2} \rfloor, \end{cases}$
(b) $\chi(P_k^m) = min\{m + 1, k\}$.

Proof. First, we prove part (a). It is easy to see that $\chi(C_k^m) = \chi(K_k) = k$ provided that $m \geq \lfloor \frac{k}{2} \rfloor$. Thus, suppose that $m < \lfloor \frac{k}{2} \rfloor$. Obviously, we can show that $\omega(C_k^m) = m + 1$. In fact, any set of $m + 1$ consecutive vertices of C_k^m is a clique of maximum size. Therefore, $\chi(C_k^m) \geq m+1$. Let $V(C_k) = \{1, 2, \ldots, k\}$ and $E(C_k) = \{\{1, 2\}, \{2, 3\}, \ldots, \{k-1, k\}, \{k, 1\}\}$. If $m+1|k$, then $\lceil \frac{k}{\lfloor \frac{k}{m+1} \rfloor} \rceil = m+1$. Hence, to have a proper coloring of C_k^m with $m+1$ colors, it's enough to color the vertices $i + j(m + 1)$ with the color i when $1 \leq i \leq m + 1$ and $0 \leq j \leq \frac{k}{m+1} - 1$. Finally, suppose that $m + 1 \nmid k$. Because any set of $m + 1$ consecutive vertices of C_k^m is a clique of maximum size, then any stable set and specially any color class of a proper coloring, has at most $\lfloor \frac{k}{m+1} \rfloor$ vertices. Therefore, $\chi(C_k^m)\lfloor \frac{k}{m+1} \rfloor \geq k$ that concludes $\chi(C_k^m) \geq \lceil \frac{k}{\lfloor \frac{k}{m+1} \rfloor} \rceil$. Now we show that $\lceil \frac{k}{\lfloor \frac{k}{m+1} \rfloor} \rceil$ colors are enough. Let $k = (m + 1)q + r$ and $1 \leq r < m + 1$. Then we have $\lceil \frac{k}{\lfloor \frac{k}{m+1} \rfloor} \rceil = m + 1 + \lceil \frac{r}{q} \rceil$. Suppose that $r = qq_1 + r_1$ and $0 \leq r_1 < q$. In what follows, we partite $V(C_k^m)$ to q subsets, such that each of these subsets contains $m + 1 + \lceil \frac{r}{q} \rceil$ or $m + 1 + \lfloor \frac{r}{q} \rfloor$ consecutive vertices and then we color the vertices of each subset with the colors $\{1, 2, \ldots, \chi\}$ consecutively.
Precisely, to have a proper coloring of C_k^m with $m + 1 + \lceil \frac{r}{q} \rceil$ colors, we consider two cases.
Case 1. If $1 \leq r_1 < q$, it's enough to color the vertices $i+j(m+1+\lceil \frac{r}{q} \rceil)$ with the color i when $1 \leq i \leq m+1+\lceil \frac{r}{q} \rceil$ and $0 \leq j \leq r_1-1$. Also, color the vertices $i+j(m+1+\lfloor \frac{r}{q} \rfloor)+r_1(m+1+\lceil \frac{r}{q} \rceil)$ with color i when $1 \leq i \leq m + 1 + \lfloor \frac{r}{q} \rfloor$ and $0 \leq j \leq q - r_1 - 1$.
Case 2. If $r_1 = 0$, then $\lfloor \frac{r}{q} \rfloor = \lceil \frac{r}{q} \rceil$ and it's enough to color the vertices $i + j(m + 1 + \lceil \frac{r}{q} \rceil)$ with the color i when $1 \leq i \leq m + 1 + \lceil \frac{r}{q} \rceil$ and $0 \leq j \leq q - 1$.
Now we prove part (b). We know that $d_{P_k}(x, y) \leq k - 1$. So if $m + 1 \geq k$, then $P_k^m = K_k$ and $\chi(C_k^m) = k$. Now suppose that $m + 1 < k$. We show that $m + 1$ colors are enough. Let $l = s(m + 1) > k + m + 1$ and consider the graph C_l^m. In view of the first part, this graph

has a proper coloring with $m+1$ colors. Because any k consecutive vertices of C_l^m induce a subgraph isomorphic to P_k^m, so any proper coloring of C_l^m gives us a proper coloring of P_k^m. ■

The following corollary is a direct consequence of Theorem 2.

Corollary 1. *Let $m, n, k \in \mathbb{N}$ and $k \geq 3$. Then*
(a) $\chi(C_k^{\frac{m}{n}}) = \begin{cases} nk & m \geq \frac{nk}{2} \\ \lceil \frac{nk}{\lfloor \frac{nk}{m+1} \rfloor} \rceil & m < \frac{nk}{2}, \end{cases}$
(b) $\chi(P_k^{\frac{m}{n}}) = min\{m+1, (k-1)n+1\}$.

Proof. Note that $C_k^{\frac{m}{n}} = C_{kn}^m$ and $P_k^{\frac{m}{n}} = P_{(k-1)n+1}^m$. Hence, in view of Theorem 2, we can easily conclude this corollary. ■

In Theorem 2 and Corollary 1 we consider the graphs with maximum degree 2. Hereafter, we focus on the graphs with maximum degree greater than 2.

Remark 1. *Let G be a connected graph and n be a positive integer greater than 1. Then easily one can see that at most three colors are enough to achieve a proper coloring of $G^{\frac{1}{n}}$. Precisely,*

$$\chi(G^{\frac{1}{n}}) = \begin{cases} 3 & n \equiv 1 (mod\, 2) \ \& \ \chi(G) \geq 3 \\ 2 & otherwise. \end{cases}$$

In Lemmas $1-4$, we construct the steps of the proof of Theorem 3 which states that $\chi(G^{\frac{2}{n}}) = \omega(G^{\frac{2}{n}})$ for any positive integer n greater than 2.

Lemma 1. *Let G be a graph, $n, m \in \mathbb{N}$ and $m < n$. If $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$, then $\chi(G^{\frac{m}{n+m+1}}) = \omega(G^{\frac{m}{n+m+1}})$.*

Proof. Note that $\omega(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n+m+1}})$. Hence, it is sufficient to show that $\chi(G^{\frac{m}{n+m+1}}) \leq \chi(G^{\frac{m}{n}})$. Let $f : V(G^{\frac{m}{n}}) \to [\chi]$ be a proper coloring of $G^{\frac{m}{n}}$ with $\chi = \chi(G^{\frac{m}{n}})$ colors. Choose the edge incident to vertex i in the hyperedge P_{ij}, for any $i < j$. Then replace each of these edges by an $(m+2)-$path, to make $G^{\frac{m}{n+m+1}}$. Now we color the new vertices on the hyperedge P_{ij}, by the set of colors $S_{ij} = \{f(v) | v \in P_{ij}\}$, such that the resulting coloring $(f' : V(G^{\frac{m}{n+m+1}}) \to [\chi])$ is a proper coloring of $G^{\frac{m}{n+m+1}}$. Precisely, suppose that an edge between i and $i(1)j$ is replaced by an $(m+2)-$path $(i, i_1, i_2, \ldots, i_{m+1}, i(1)j)$. Now use the color of $i(k)j$ to color the new vertex i_k when $1 \leq k \leq m+1$. Easily we can show that this coloring is a proper coloring. Consider $f'(N_m(i_k) \setminus \{i_k\})$ that is the colors of all vertices with distance $d \in \{1, 2, \ldots, m\}$ from i_k. Because $f'(i_l) = f(i(l)j)$ so $f'(N_m(i_k) \setminus \{i_k\}) \subseteq f(N_m(i(k)j) \setminus \{i(k)j\})$. Then the

color of i_k is different from the color of any other vertex in $N_m(i_k)$. In addition, for any two vertices x and y of $G^{\frac{m}{n}}$ with $f(x) = f(y)$, we have

$$d_{G^{\frac{m}{n+m+1}}}(x, y) \geq d_{G^{\frac{m}{n}}}(x, y) > m.$$

Hence, the coloring f' is a proper coloring for $G^{\frac{m}{n+m+1}}$. ■

Lemma 2. *Let G be a connected graph of order n. Then there is a labeling $f : V(G) \to [n]$, such that,*
(a) for any two distinct vertices i and j of G, $f(i) \neq f(j)$ and
(b) for any vertex x, if $f(x) \neq n$, then $f(x) < f(y)$ for some vertex $y \in N(x)$.

Proof. A labeling f of $V(G)$ is called a *coloring order* of G, if it satisfies two properties (a) and (b). The proof is by induction on n. Obviously, the lemma is correct for $n = 1$. Assume that there is a coloring order for any connected graph of order $n-1$. Because G is connected, so there is a vertex x, such that $G[V(G) \setminus \{x\}]$ is connected. Then $G[V(G) \setminus \{x\}]$ has a coloring order like $f : G[V(G) \setminus \{x\}] \to [n-1]$. Now we can define a coloring order for G as follows:

$$f'(i) = \begin{cases} 1 & i = x \\ f(i) + 1 & i \neq x. \end{cases}$$

Clearly one can check that f' is a coloring order for G. ■

Lemma 3. *Let G be a connected graph with $\Delta(G) \geq 3$. Then $\chi(G^{\frac{2}{3}}) = \Delta(G) + 1 = \omega(G^{\frac{2}{3}})$.*

Proof. It is well-known that any connected graph with maximum degree Δ is a subgraph of a connected $\Delta-$regular graph. Hence, it is sufficient to prove the lemma for the connected $\Delta-$regular graph G of order n.
Color all t-vertices by the color 0. We show that the colors of $C = \{1, 2, \ldots, \Delta\}$ are enough to color i-vertices. Suppose that f is a coloring order of G and $f^{-1}(i)$ is denoted by f_i . We color the i-vertices of $N(f_1)$, $N(f_2)$, $\ldots$ and $N(f_n)$ consecutively in n steps:
Step 1. We color optionally the i-vertices of $N(f_1)$ with the colors of C.
Step 2. If two vertices f_1 and f_2 are not adjacent in G, then we can color optionally the i-vertices of $N(f_2)$. If f_1 and f_2 are adjacent in G, then at first we color the i-vertex $f_2(1)f_1$, with color c_1 which is different from the color of $f_1(1)f_2$. Then we color the other i-vertices of $N(f_2)$ with the colors of $C \setminus \{c_1\}$ optionally.
Step k $(3 \leq k \leq n-1)$. By considering the colored vertices in previous steps, there are Δ or $\Delta - 1$ colors to color each vertex of $N(f_k)$. Now by using Hall's Theorem [4, page 419], we can find a perfect matching between the vertices of $N(f_k)$ and the colors $C = \{1, 2, \ldots, \Delta\}$ which leads us to a coloring of the vertices of $N(f_k)$.

Step n. Let $N_G(f_n) = \{f_{i_1}, f_{i_2}, \ldots, f_{i_\Delta}\}$. Suppose that c_j is the color of $f_{i_j}(1)f_n$, for any $1 \le j \le \Delta$. If $|\{c_j | 1 \le j \le \Delta\}| = k \ge 2$, then we select k vertices of

$$\{f_{i_1}(1)f_n, f_{i_2}(1)f_n, \ldots, f_{i_\Delta}(1)f_n\}$$

with different colors and color their neighbors in $N(f_n)$, with a disarrangement of that k colors. After that, we can color the remaining i-vertices of $N(f_n)$, with the colors of $C \setminus \{c_1, c_2, \ldots, c_\Delta\}$.

Now let $k = 1$. In other words, assume that $c_1 = c_2 = \ldots = c_\Delta = a$.

To color the i-vertices of $N(f_n)$ in this case, we need to change the color of one i-vertex of $F = \{f_{i_1}(1)f_n, f_{i_2}(1)f_n, \ldots, f_{i_\Delta}(1)f_n\}$.

Consider the subgraph of $G^{\frac{2}{3}}$, induced by the vertices of colors a and b and let H be the connected component of this subgraph, containing the vertex $f_{i_1}(1)f_n$. Clearly, we have $1 \le d_H(x) \le 2$ for each $x \in V(H)$ and $d_H(f_{i_1}(1)f_n) = 1$. Therefore, H is a path and we can interchange k to 2, by changing the colors a and b in H and then we can color the i-vertices of $N(f_n)$. ■

Lemma 4. *Let G be a connected graph with $\Delta(G) \ge 3$. Then*
(a) $\chi(G^{\frac{2}{4}}) = \Delta(G) + 1 = \omega(G^{\frac{2}{4}})$,
(b) $\chi(G^{\frac{2}{5}}) = \Delta(G) + 1 = \omega(G^{\frac{2}{5}})$.

Proof. First, we prove part (a). We use the proper coloring of $G^{\frac{2}{3}}$ which is defined in Lemma 3. Suppose that $f : V(G^{\frac{2}{3}}) \to C$ is a proper coloring of $G^{\frac{2}{3}}$ where $C = \{0, 1, 2, \ldots, \Delta(G)\}$. Three colors are appeared on each hyperedge of $G^{\frac{2}{3}}$. Because $\Delta(G)+1 \ge 4$, so for each hyperedge P_{ij}, there is at least one color, denoted by c_{ij}, which is not used for the coloring of the vertices of P_{ij}. Now on each hyperedge P_{ij}, replace the edge between $i(1)j$ and $j(1)i$ by a $2-$path and color the new vertex by c_{ij}. In other words, we have a proper coloring f' for $G^{\frac{2}{4}}$ as follows:

$$f'(x) = \begin{cases} c_{ij} & x = i(2)j = j(2)i \\ f(x) & otherwise. \end{cases}$$

Clearly, we can show that f' is a proper coloring for $G^{\frac{2}{4}}$.

Now, we prove part (b). Let $f : E(G) \to C$ be a proper edge coloring of G, where $C = \{0, 1, 2, \ldots, \Delta(G)\}$. At first, we color the i-vertices $i(1)j$ and $j(1)i$ with the same color of $f(ij)$. Then we color all t-vertices of $G^{\frac{2}{5}}$. Color the t-vertex i, with one color of $C \setminus \{f(ij) | d_G(j,i) = 1\}$ which is non-empty. Finally we color two remaining i-vertices of each hyperedge. There are two cases for each hyperedge P_{ij}:

Case 1. Two vertices i and j have the same color.

In this case, only two colors are used on the hyperedge P_{ij}. Because $\Delta(G)+1 \geq 4$, so there are at least two colors which are not appeared on the vertices of P_{ij}, and we can color two uncolored i-vertices of P_{ij}, with the remaining colors.

Case 2. Two vertices i and j have different colors.

In this case, we color the i-vertex $i(2)j$ with the color of j and the i-vertex $j(2)i$ with the color of i. Note that $d_{G^{\frac{2}{5}}}(i(2)j, j) = 2 = d_{G^{\frac{2}{5}}}(j(2)i, i)$ and $d_{G^{\frac{2}{5}}}(i(2)j, i) = 1 = d_{G^{\frac{2}{5}}}(j(2)i, j)$. Clearly, one can show that this coloring is a proper coloring of $G^{\frac{2}{5}}$. ■

Now we can prove the following theorem inductively by applying Theorem 1 and Lemmas 1, 3 and 4.

Theorem 3. *Let G be a connected graph with $\Delta(G) \geq 3$ and n be a positive integer greater than 2. Then $\chi(G^{\frac{2}{n}}) = \Delta(G) + 1 = \omega(G^{\frac{2}{n}})$.*

Here, we show that the conjecture A is true for some rational number $\frac{m}{n} < 1$.

Theorem 4. *Let G be a connected graph with $\Delta(G) \geq 3$ and $m \in \mathbb{N}$. Then $\chi(G^{\frac{m}{m+1}}) = \omega(G^{\frac{m}{m+1}})$.*

Proof. We prove this theorem by induction on m. Remark 1 and Lemma 3 show that the assertion holds for $m = 1, 2$. We prove that if the assertion holds for $m = 2k \in \mathbb{N}$, then it is correct for $m = 2k+1$ and $m = 2k+2$. Suppose that $\chi(G^{\frac{2k}{2k+1}}) = \omega(G^{\frac{2k}{2k+1}}) = k\Delta(G)+1$. To prove $\chi(G^{\frac{2k+1}{2k+2}}) = k\Delta(G) + 2$, we follow the same lines as in the proof of Lemma 4. Assume that $f : V(G^{\frac{2k}{2k+1}}) \to C$ is a proper coloring of $G^{\frac{2k}{2k+1}}$ where $C = \{0, 1, 2, \ldots, k\Delta(G)\}$. Then we can define a proper coloring $f' : V(G^{\frac{2k+1}{2k+2}}) \to C \cup \{k\Delta(G) + 1\}$ as follows:

$$f'(x) = \begin{cases} k\Delta(G)+1 & x = i(k+1)j \\ f(x) & x = i(l)j,\ l \leq k. \end{cases}$$

In fact, one i-vertex with the color $k\Delta(G) + 1$, is added to each hyperedge P_{ij} of $G^{\frac{2k}{2k+1}}$ between two central i-vertices of P_{ij} to construct $G^{\frac{2k+1}{2k+2}}$. We show that f' is a proper coloring for $G^{\frac{2k+1}{2k+2}}$.

Suppose that x and y are two adjacent vertices of $G^{\frac{2k+1}{2k+2}}$. Therefore, $d_{G^{\frac{1}{2k+2}}}(x, y) \leq 2k + 1$. It is easy to check that $f'(x) \neq f'(y)$ when $f'(x) = k\Delta(G)+1$ or $f'(y) = k\Delta(G)+1$. Hence, we assume that $f'(x) \neq k\Delta(G) + 1 \neq f'(y)$. If $d_{G^{\frac{1}{2k+2}}}(x, y) \leq 2k$, then before adding new i-vertices, we must have $d_{G^{\frac{1}{2k+1}}}(x, y) \leq 2k$. Then x and y are adjacent in $G^{\frac{2k}{2k+1}}$ and their colors are different in f. Now suppose that $d_{G^{\frac{1}{2k+2}}}(x, y) = 2k + 1$. Therefore, there is a path of length $2k + 1$ between x and y which contains a new i-vertex with the color $k\Delta(G) + 1$. Hence, the distance of x to y in $G^{\frac{1}{2k+1}}$ is at most $2k$. Thus, x and y are adjacent in $G^{\frac{2k}{2k+1}}$

and their colors are different. Consequently, f' is a proper coloring for $G^{\frac{2k+1}{2k+2}}$.
To prove $\chi(G^{\frac{2k+2}{2k+3}}) = (k+1)\Delta(G)+1$, suppose that $f : V(G^{\frac{2k}{2k+1}}) \to C$ is a proper coloring of $G^{\frac{2k}{2k+1}}$ such that $C = \{0, 1, 2, \ldots, k\Delta(G)\}$ and all t-vertices, have the same color 0. To construct $G^{\frac{2k+2}{2k+3}}$ from $G^{\frac{2k}{2k+1}}$, subdivide the central edge $\{i(k)j, j(k)i\}$ of any hyperedge P_{ij} to three edges $\{i(k)j, i[k+1]j\}$, $\{i[k+1]j, i[k+2])j\}$ and $\{i[k+2]j, j(k)i\}$ which contain two new i-vertices $i[k+1]j$ and $i[k+2]j$. Now let S be the set of the new i-vertices and all t-vertices. Clearly we can show that $G^{\frac{2k+2}{2k+3}}[S]$ is isomorphic to $G^{\frac{2}{3}}$. Thus, in view of Lemma 3, we have $\chi(G^{\frac{2k+2}{2k+3}}[S]) = \Delta(G)+1$. Now let $f' : V(G^{\frac{2k+2}{2k+3}}[S]) \to \{0, c_1, c_2, \ldots, c_\Delta\}$ be a proper coloring of $G^{\frac{2k+2}{2k+3}}[S]$ such that the color of all t-vertices is 0 and $\{0, c_1, c_2, \ldots, c_\Delta\} \cap C = \{0\}$. Now we can define a proper coloring f'' for $G^{\frac{2k+2}{2k+3}}$, as follows:

$$f''(x) = \begin{cases} f'(x) & x \in S \\ f(x) & x = i(l)j,\, l \leq k. \end{cases}$$

Suppose that x and y are two adjacent vertices of $G^{\frac{2k+2}{2k+3}}$. If $x, y \in S$, then they are adjacent in $G^{\frac{2k+2}{2k+3}}[S]$. Hence, $f''(x) = f'(x) \neq f'(y) = f''(y)$. If only one of them is in S, obviously we have $f''(x) = f'(x) \neq f''(y)$ or $f''(y) = f'(y) \neq f''(x)$. Now suppose that $x, y \in V(G^{\frac{2k+2}{2k+3}}) \setminus S$. Therefore, $d_{G^{\frac{1}{2k+3}}}(x, y) \leq 2k+2$. If $d_{G^{\frac{1}{2k+3}}}(x, y) \leq 2k$, then before adding central two i-vertices to each hyperedge, we have $d_{G^{\frac{1}{2k+1}}}(x, y) \leq 2k$. Then x and y are adjacent in $G^{\frac{2k}{2k+1}}$ and their colors are different. Now suppose that $d_{G^{\frac{1}{2k+3}}}(x, y) \geq 2k+1$. Therefore, there is a path of length $2k+1$ or $2k+2$ between x and y, which contains two new i-vertices of S. Hence, the distance of x to y in $G^{\frac{1}{2k+1}}$ is at most $2k$. Thus, x and y are adjacent in $G^{\frac{2k}{2k+1}}$ and their colors are different. ■

Corollary 2. *Let G be a connected graph with $\Delta(G) \geq 3$ and $k, m \in \mathbb{N}$. Then $\chi(G^{\frac{m}{k(m+1)}}) = \omega(G^{\frac{m}{k(m+1)}})$.*

Proof. We can result this corollary by using Lemma 1 and Theorem 4. ■

Theorem 5. *Let G be a connected graph and $1 < m \in \mathbb{N}$.*
(a) If m is an even integer and $\Delta(G) \geq 4$, then for any integer $n \geq 2m+2$

$$\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}}).$$

(b) If m is an odd integer and $\Delta(G) \geq 5$, then for any integer $n \geq 2m+2$

$$\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}}).$$

Proof. The proofs of parts (a) and (b) are similar. Thus, we only prove the first part.
The proof is by induction on n. Corollary 2 shows that (a) is holds for $n = 2m+2$. Suppose

that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ for some $n \geq 2m+2$. Let $f : V(G^{\frac{m}{n}}) \rightarrow [\chi]$ be a proper coloring of $G^{\frac{m}{n}}$ when $\chi = \chi(G^{\frac{m}{n}})$. We extend this coloring to a proper coloring of $G^{\frac{m}{n+1}}$. On each hyperedge P_{ij}, Consider $2m$ consecutive i-vertices $i(1)j, i(2)j, \ldots, i(2m)j$ and let $V_{ij} = \{i(1)j, i(2)j, \ldots, i(2m)j\}$. So $|f(V_{ij})| \leq 2m$.
We have $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}}) = \frac{m}{2}\Delta(G) + 1 \geq 2m+1$. Thus, for any hyperedge P_{ij}, there is a color which does not assign to the vertices of V_{ij}, denoted by c_{ij}. Subdivide the edge $\{i(m)j, i(m+1)j\}$ of each hyperedge P_{ij}, to two edges and color the new i-vertex by c_{ij}. It is easy to check that we have a proper coloring of $G^{\frac{m}{n+1}}$. ■

We can divide Conjecture A to the following conjectures.
Conjecture $A(m)$***.*** *Let G be a connected graph with $\Delta(G) \geq 3$ and m be a positive integer greater than 1. Then for any positive integer $n \geq m$, we have $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*
In view of Theorem 3, Conjecture $A(2)$ holds. In addition, by applying Lemma 1, we can show that $A(m)$ holds if it is correct only for any $n \in \{m+1, m+2, \ldots, 2m+1\}$. The Conjectures A and $A(m)$ remain open for any $m \geq 3$.
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